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Critical Exponents for Numbers of Differently Anchored Polymer Chains on Fractal Lattices with Adsorbing Boundaries

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We study the problem of polymer adsorption in a good solvent when the container of the polymer–solvent system is taken to be a member of the Sierpinski gasket (SG) family of fractals. Members of the SG family are enumerated by an integer b ($2 \leqslant b \leqslant \infty$), and it is assumed that one side of each SG fractal is an impenetrable adsorbing boundary. We calculate the critical exponents γ_1, γ_{11}, and γ_s, which, within the self-avoiding walk model (SAW) of the polymer chain, are associated with the numbers of all possible SAWs with one, both, and no ends anchored to the adsorbing impenetrable boundary, respectively. By applying the exact renormalization group (RG) method for $2 \leqslant b \leqslant 8$ and the Monte Carlo renormalization group (MCRG) method for a sequence of fractals with $2 \leqslant b \leqslant 80$, we obtain specific values for these exponents. The obtained results show that all three critical exponents γ_1, γ_{11}, and γ_s, in both the bulk phase and crossover region are monotonically increasing functions with b. We discuss their mutual relations, their relations with other critical exponents pertinent to SAWs on the SG fractals, and their possible asymptotic behavior in the limit $b \to \infty$, when the fractal dimension of the SG fractals approaches the Euclidean value 2.

KEY WORDS: Polymer adsorption; fractals; exact and Monte Carlo renormalization group.

1. INTRODUCTION

Polymer adsorption on a rigid impenetrable adsorbent has been widely studied because of its practical and theoretical importance. The statistical

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mechanics approach to this problem has been successfully applied, in particular, in the case of a good solvent that contains only one linear polymer interacting with the impenetrable wall (boundary). More specifically, because of the successful recognition of the polymer adsorption problem as a surface critical phenomenon, it has been possible to express various polymer quantities in terms of power laws with the concomitant critical exponents. Motivated by the relevant work done in the case of Euclidean lattices, several studies have appeared recently(1-6) in which it has been assumed that the adsorbate is immersed in a fractal container. In these studies, almost exclusively, only two critical exponents have been studied, that is, the end-to-end distance critical exponent ν and the crossover exponent ϕ. However, a thorough picture of the adsorption problem requires knowledge of critical exponents that describe numbers of polymer configurations grouped according to the different ways of anchoring to the adsorbing boundary. In terms of the self-avoiding random walk (SAW) model of linear polymers, these exponents are defined by the following formulas for numbers of possible different configurations averaged over the number of sites on the impenetrable boundary:

$$C_1(N, T) \sim \mu^N N^{\gamma_1 - 1}, \qquad C_{11}(N, T) \sim \mu^N N^{\gamma_{11} - 1}, \qquad C_s(N, T) \sim \mu^N N^{\gamma_s - 1} \tag{1}$$

which are assumed to be valid for a large number N of steps (monomers). Here C_1, C_{11}, and C_s are the numbers of all possible SAWs with one, both, and no ends anchored to the boundary, respectively, while γ_1, γ_{11}, and γ_s are the respective critical exponents. It is assumed that the numbers C_1, C_{11}, and C_s are functions of temperature T through the connectivity constant μ. It turns out that μ is a continuous function of T (see, for instance, ref. 1), so that in the high-temperature region, that is, in the bulk phase, μ is constant up to the crossover temperature T_a, whereas for $T < T_a$ (in the adsorbed phase) it is a monotonically increasing function.

In this paper we calculate the critical exponents γ_1, γ_{11}, and γ_s for the SAW model of a polymer chain situated on fractals that belong to the Sierpinski gasket (SG) family. Each member of the SG family is labeled by an integer b ($2 \leqslant b \leqslant \infty$), and it is assumed that one side of each SG fractal is an impenetrable adsorbing wall. The exponents γ_1, γ_{11}, and γ_s have been known(2) only in the case $b = 2$ for the bulk phase and crossover region. In this work, we calculate these critical exponents by applying the exact renormalization group (RG) method for $2 \leqslant b \leqslant 8$ and the Monte Carlo renormalization group (MCRG) method for a sequence of fractals with $2 \leqslant b \leqslant 80$. The obtained results show that all three critical exponents (in both the bulk phase and crossover region) are monotonically increasing functions with b, which should be associated with the fact$^{(7,8)}$ that the critical exponent γ (which determines the asymptotic behavior of the total number of distinct SAWs) for the SG fractals is also a monotonically increasing function of b, with the specific behavior$^{(9)}$ in the asymptotic region $b \to \infty$.

This paper is organized as follows. In Section 2 we present the general framework of the RG method for studying SAWs on the SG fractals in a way that should make the method transparent for exact calculations as well as for the Monte Carlo (MC) calculations of the SAW critical exponents. In Section 3 we present the exact results for the critical exponents γ_1, γ_{11}, and γ_s for $2 \leqslant b \leqslant 8$ and the MCRG results for a sequence of the SG fractals up to $b \leqslant 80$. In the same section we discuss the obtained results and their relevance to the current knowledge of statistics of SAWs on fractals.

2. FRAMEWORK OF THE RENORMALIZATION GROUP APPROACH

In this section we expound on the RG approach of calculating the critical exponents γ_1, γ_{11}, and γ_s for the SAW adsorption problem on the SG family of fractals. These fractals have been studied in numerous papers (see, for instance, ref. 7), and consequently we shall give here only a brief summary of their basic properties. We start by recalling the fact that each member of the SG fractal family can be constructed in stages. At the initial stage ($r = 1$) of the construction there is an equilateral triangle (generator) that contains b^2 identical smaller triangles of unit side length, out of which only the upper oriented ones are physically present. The subsequent fractal stages are constructed self-similarly, so that the complete fractal is obtained in the limit $r \to \infty$. In the case under study, it is assumed that one side of each fractal is an impenetrable attractive wall, and, for the sake of convenience, we assume that it is the base of the corresponding triangle. Thus it follows that the fractal dimension of the adsorbing wall (surface) is $d_s = 1$, whereas the fractal dimension of the complete SG fractal is known to be $d_f = \ln[b(b+1)/2]/\ln b$.

In order to study influence of the adsorbing wall on the polymer statistics, we introduce the two Boltzmann factors $w = e^{-\varepsilon_w/T}$ and $t = e^{-\varepsilon_t/T}$, where ε_w is the energy of a monomer lying on the adsorbing wall and ε_t is the energy of a monomer that appears in the layer adjacent to the wall. Here we set the Boltzmann constant k_B equal to unity. We should also set $\varepsilon_t > 0$ so as to prevent the tendency of a polymer chain toward being always adsorbed.$^{(1)}$ In the parlance appropriate for the SAW model, we assign the weight x to each step in the bulk (away from the wall), the weight wx to each step on the wall, and the weight tx to each step in the layer adjacent to the wall.

The weighting factors defined in the foregong paragraph allow us to introduce the requisite generating functions together with their assumed asymptotic forms:

$$\begin{aligned} C_1(x, T) &= \sum_{N=1}^{\infty} x^N \sum_{M,L=1}^{N} C_1(N, M, L)\, w^M t^L \\ &= \sum_{N=1}^{\infty} C_1(N, T)\, x^N \sim (1 - x\mu)^{-\gamma_1} \end{aligned} \tag{2}$$

$$\begin{aligned} C_{11}(x, T) &= \sum_{N=1}^{\infty} x^N \sum_{M,L=1}^{N} C_{11}(N, M, L)\, w^M t^L \\ &= \sum_{N=1}^{\infty} C_{11}(N, T)\, x^N \sim (1 - x\mu)^{-\gamma_{11}} \end{aligned} \tag{3}$$

$$\begin{aligned} C_s(x, T) &= \sum_{N=1}^{\infty} x^N \sum_{M,L=1}^{N} C_s(N, M, L)\, w^M t^L \\ &= \sum_{N=1}^{\infty} C_s(N, T)\, x^N \sim (1 - x\mu)^{-\gamma_s} \end{aligned} \tag{4}$$

where $C_1(N, M, L)$ $[C_{11}(N, M, L)]$ represents the number of N-step SAWs with M steps on the surface and L steps in the layer adjacent to the wall provided one (both) end(s) of the walk is (are) anchored to the wall, while $C_s(N, M, L)$ is the number of SAWs with no ends anchored to the wall. The power laws at the ends of the above equations represent leading singular behaviors of the corresponding generating functions, which are valid when x approaches $x_0 = 1/\mu(T)$ from below.

To calculate the critical exponents, we have found that it is helpful to define, for an SG triangle at the rth stage of construction, 20 restricted partition functions(2) (see Fig. 1) that provide a complete description of the above generating functions. In other words, we can express each generating function in terms of the restricted partition functions. It is convenient to start with the generating function C_{11}, which can be written in the form

$$C_{11}(x, T) = \sum_{r=1}^{\infty} \frac{F_{11}^{(r)}}{b^r} \tag{5}$$

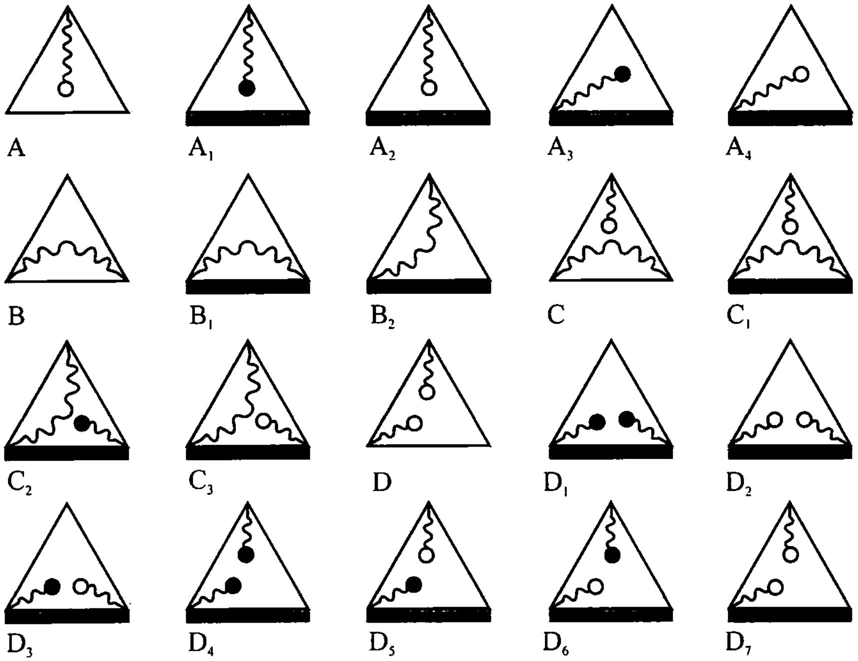

Fig. 1. Diagrams representing the 20 restricted partition functions for SAWs on an SG triangle at the rth stage of construction. The solid circles denote ends of SAWs that are anchored to the attractive boundary, while the open circles are those ends that are somewhere in the bulk.

with

$$F_{11}^{(r+1)} = f_1 A_1^2 + f_2 A_1 A_3 + f_3 A_3^2 + f_4 A_1 C_2 + f_5 A_3 C_2 + f_6 C_2^2 + f_7 D_1 + f_8 D_4 + f_9 A_1 + f_{10} A_3 + f_{11} C_2 \tag{6}$$

where we have suppressed the superscript (r) on the right-hand side of the above relation. Here, the coefficients f_i are polynomials of $B^{(r)}$, $B_1^{(r)}$, and $B_2^{(r)}$. Similarly, the generating functions $C_1(x, T)$ and $C_s(x, T)$ have the form

$$C_1(x, T) = C_{11}(x, T) + \sum_{r=1}^{\infty} \frac{F_1^{(r)}}{b^r}, \qquad C_s(x, T) = \sum_{r=1}^{\infty} \frac{F_s^{(r)}}{b^r} \tag{7}$$

where $F_1^{(r+1)}$ is a quadratic function of $A^{(r)}$, $A_i^{(r)}$, $C^{(r)}$, and $C_i^{(r)}$ and is linear in $D_3^{(r)}$, $D_5^{(r)}$, and $D_6^{(r)}$, whereas $F_s^{(r+1)}$ is a quadratic function of $A^{(r)}$, $A_2^{(r)}$, $A_4^{(r)}$, $C^{(r)}$, $C_1^{(r)}$, and $C_3^{(r)}$, and is linear in $D^{(r)}$, $D_2^{(r)}$, and $D_7^{(r)}$, with the corresponding coefficients being polynomials of $B^{(r)}$, $B_1^{(r)}$, and $B_2^{(r)}$.

For arbitrary r, the self-similarity of the fractals under study implies the following recursion relations for the bulk restricted partition functions:

$$B' = f_b, \qquad A' = a_1 A + a_2 C, \qquad C' = c_1 A + c_2 C \tag{8}$$

where we have used the prime for the $(r+1)$th-order functions and no indices for the rth-order functions, while f_b, a_i, and c_i are polynomials in B (whose explicit forms for $2 \leqslant b \leqslant 8$ have been found$^{(7)}$). Analogously, we find the additional recursion relations

$$v'_1 = \hat{\mathbf{U}} v_1, \qquad v'_2 = \hat{\mathbf{U}} v_2 + \hat{\mathbf{V}} v_3 \tag{9}$$

$$\begin{aligned} B'_1 &= g_1 B_1^2 + g_2 B_1 B_2 + g_3 B_2^2 \\ B'_2 &= g_4 B_1 + g_5 B_2 \\ C'_1 &= g_6 C_1 + g_7 A + g_8 C \end{aligned} \tag{10}$$

where v_1, v_2, and v_3 are vector columns with components (A_1, A_3, C_2), (A_2, A_4, C_3), and (C_1, A, C), respectively, and elements of matrices $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$, as well as the coefficients g_i, are polynomials of $B^{(r)}$, $B_1^{(r)}$, and $B_2^{(r)}$. The remaining restricted partition functions D and D_i ($i = 1,..., 7$) satisfy similar recursion relations, but their explicit forms are somewhat more intricate, and, in addition, it turns out that they are not needed for the determination of the critical exponents. For this reason, we do not give them here.

The preceding recursion relations comprise the RG transformations of the problem studied, and, in accord with the accepted physical picture of the interaction parameters, we assume the following initial conditions:

$$\begin{gathered} B^{(0)} = x, \qquad B_1^{(0)} = wx, \qquad B_2^{(0)} = tx \\ A^{(0)} = C^{(0)} = D^{(0)} = A_i^{(0)} = C_i^{(0)} = D_i^{(0)} = 0 \end{gathered} \tag{11}$$

which are pertinent to the unit SG triangle $(r = 0)$.

The numerical study of the recursion relations for B [given by (8)], B_1, and B_2 [given by (10)], with the respective initial conditions (11), shows that, for any fixed value of $t < 1$, there are three different temperature regions,$^{(1,2,5)}$ which we discuss separately, starting with the high-temperature region.

(i) At high temperatures, that is, for $w < w^*(t)$, the critical fugacity is constant and equal to its bulk critical value $x_c = 1/\mu$. For all these values of temperature the bulk SAW fixed point is reached

$$(B^*, B_1^*, B_2^*) = (B^*, 0, 0) \tag{12}$$

The fraction of SAW steps in contact with the surface vanishes in this temperature region, so that the polymer is in the desorbed state. Linearization about this fixed point leads to only one relevant eigenvalue

$$\lambda = \left. \frac{\partial B'}{\partial B} \right|_{B=B^*} \tag{13}$$

which yields the value of the end-to-end distance critical exponent $\nu = \ln b / \ln \lambda$.

In order to calculate exponents γ_1, γ_{11}, and γ_s, it is necessary to investigate the singular behavior of the generating functions (5) and (7), which requires that we analyze the restricted partition functions (8)–(10) in the vicinity of the bulk fixed point (12). To this end, we choose[7,10] a small positive number ε and a value of x sufficiently close to x_c so that $x_c - x \ll \varepsilon \ll 1$. Then, for $r < r_0 = \ln[\varepsilon/(x_c - x)]/\ln \lambda \gg 1$, we find

$$\begin{array}{llll} B^{(r)} \approx B^*, & B_1^{(r)} \approx 0, & B_2^{(r)} \sim q^r & \\ A^{(r)} \sim \lambda_1^r, & A_1^{(r)} \sim \lambda_2^r, & A_2^{(r)} \sim \lambda_1^r, & A_3^{(r)} \approx \text{const}, \\ A_4^{(r)} \sim (q\lambda_1)^r & & & \\ C^{(r)} \sim \lambda_1^r, & C_1^{(r)} \approx 0, & C_2^{(r)} \approx 0, & C_3^{(r)} \sim (q^2\lambda_1)^r \end{array} \tag{14}$$

where, on the grounds of Eq. (8), λ_1 is the relevant eigenvalue of the matrix[7]

$$\begin{pmatrix} a_1(B^*) & a_2(B^*) \\ c_1(B^*) & c_2(B^*) \end{pmatrix} \tag{15}$$

and

$$q = g_5(B^*, 0, 0) < 1, \qquad \lambda_2 = (\hat{\text{U}})_{11}\,(B^*, 0, 0) < \lambda_1 \tag{16}$$

For $r > r_0$, the partition functions rapidly appraoch some constants, so, analyzing the singular behavior of the generating functions (5) and (7) in accord with their assumed asymptotic forms (2)–(4), we finally get

$$\gamma_1 = \frac{\ln(\lambda_1 \lambda_2 / b)}{\ln \lambda}, \qquad \gamma_{11} = \frac{\ln(\lambda_2^2 / b)}{\ln \lambda}, \qquad \gamma_s = \frac{\ln(\lambda_1^2 / b)}{\ln \lambda} \tag{17}$$

(ii) When the temperature is lowered, an adsorption transition occurs for $w = w^*(t)$. In that case $x_c(w^*)$ is still equal to its bulk value, but Eqs. (8) and (10) iterate toward a new fixed point

$$(B^*, B_1^*, B_2^*) = (B^*, B^*, B^*) \tag{18}$$

which corresponds to the "special" transition when a balance between the attractive polymer–surface potential and an effective "entropic" repulsion sets in.(1)

The bulk restricted partition functions A, B, and C (see Fig. 1) behave in the same way as in the high-temperature region, while the leading singular behaviors of the surface restricted partition functions, in the vicinity of the special fixed point, are

$$\begin{aligned} &A_1^{(r)} \sim \lambda_3^r, \quad A_3^{(r)} \sim \lambda_3^r, \quad C_2^{(r)} \sim \lambda_3^r, \quad A_2^{(r)} \sim \lambda_1^r, \\ &A_4^{(r)} \sim \lambda_1^r, \quad C_1^{(r)} \sim \lambda_1^r, \quad C_3^{(r)} \sim \lambda_1^r \end{aligned} \tag{19}$$

where λ_3 is the largest eigenvalue of the matrix $\hat{\mathbf{U}}$, whose elements are evaluated at the symmetric fixed point (18). Using the same approach, we find

$$\gamma_1 = \frac{\ln(\lambda_1 \lambda_3/b)}{\ln \lambda}, \qquad \gamma_{11} = \frac{\ln(\lambda_3^2/b)}{\ln \lambda}, \qquad \gamma_s = \frac{\ln(\lambda_1^2/b)}{\ln \lambda} \tag{20}$$

(iii) In the low-temperature region, that is, for $w > w^*(t)$, the critical fugacity $x_c(w)$ is a decreasing function of w, while the recursion relations (8) and (10) iterate toward the fixed point

$$(B^*, B_1^*, B_2^*) = (0, 1, 0) \tag{21}$$

which corresponds to an adsorbed polymer that displays features of a one-dimensional system, so that all the critical exponents are equal to the corresponding value of γ, i.e., $\gamma_1 = \gamma_{11} = \gamma_s = 1$.

3. RESULTS AND DISCUSSION

To obtain specific values of the critical exponents γ_1, γ_{11}, and γ_s, one needs to know the eigenvalues λ, λ_1, λ_2, and λ_3. In previous studies, the exact values(2,7) and the MCRG values(8,11) for λ and λ_1 have been found for $2 \leqslant b \leqslant 8$ and for $2 \leqslant b \leqslant 80$, respectively. An exact calculation of the remaining two eigenvalues, λ_2 and λ_3, requires knowledge of the elements (which are polynomials) of the matrix $\hat{\mathbf{U}}$. They can be calculated by enumerating the SAWs that are described by the restricted partition functions A_1, A_3, and C_2 (see Fig. 1). We have found that this enumeration is feasible for $2 \leqslant b \leqslant 8$. The obtained exact values for the critical exponents relevant to the bulk phase described by the fixed point (12) together with those relevant to the crossover region described by the fixed point (18) are given in Table I.

For a sequence of $b \geqslant 9$, the exact determination of elements of the matrix $\hat{\mathrm{U}}$ hardly can be reached using present-day computers. However, to calculate λ_2 and λ_3 one does not need a complete knowledge of these polynomials (that is, one does not need to know all their coefficients). In fact, for obtaining λ_2, one needs only value of the polynomial $(\hat{\mathrm{U}})_{11}$ at the fixed point (12), whereas to obtain λ_3 one needs values of all elements of the matrix $\hat{\mathrm{U}}$ at the fixed point (18). On the other hand, the elements of the matrix $\hat{\mathrm{U}}$ can be conceived as grand partition functions of appropriate ensembles, and consequently within the MCRG method(8,12) the requisite values of these polynomials can be determined directly. Details of the way to ascertain these values are quite similar to the way applied previously(8,12) and we are not going to elaborate on it further. Our present MCRG findings for $2 \leqslant b \leqslant 80$ are given in Table I together with the exact results for $2 \leqslant b \leqslant 8$. Comparing the data given in the Table I and taking into account the known values(7,8) for γ, one can conclude that, in the region of b studied, the following inequality is valid: $\gamma_{11} < \gamma_1 < \gamma$. This result is quite plausible since γ is related to all possible SAWs, which certainly outnumber the SAWs with one or both ends anchored to the boundary, described by the power laws (1). Similarly, the generating function C_{11} describes SAWs with a stronger constraint than the constraint for SAWs described by the generating function C_1 [see (1)], and consequently $\gamma_{11} < \gamma_1$. One can also verify that data from Table I satisfy the scaling relation(2,13)

$$\gamma_s = 2\gamma_1 - \gamma_{11} = \gamma + \nu(d_f - d_s) \tag{22}$$

Inspired by this piece of information verified for $2 \leqslant b \leqslant 80$, one can look at the general formulas (17) and (20) for γ_1, γ_{11}, and γ_s as well as at the formulas(7) for γ and ν, and thereby one can conclude that the scaling relation (22) stays valid for arbitrary b in both the bulk phase and the crossover region.

For the sake of a better assessment of the global behavior of the critical exponents as functions of the scaling parameter b, we present the results in Fig. 2. Notice that both γ_1 and γ_{11}, relevant to the bulk phase of the polymer system, are monotonically increasing functions of b, so that for $b > 3$ the exponent γ_1 is larger than the corresponding two-dimensional Euclidean value(14,15) $\gamma_1 = 61/64$, whereas γ_{11} surpasses the Euclidean value(14,15) $\gamma_{11} = -3/16$ for $b > 2$. In the case of the crossover region, described by the fixed point (18), γ_1 and γ_{11} are also monotonically increasing functions of b. The behavior of the critical exponent γ_s (whose values are the same in both the bulk phase and the crossover region) is similar to the behavior of γ_1 and γ_{11}, that is, γ_s is a monotonically increasing function of b, which, for $b > 6$, surpasses its Euclidean counterpart(13) $\gamma_s = 67/32$.

Table I. The Exact RG ($2 \leqslant b \leqslant 8$) and the MCRG ($2 \leqslant b \leqslant 80$) Results for the Critical Exponents γ_1, γ_{11}, and γ_s, for the Bulk Phase ($T > T_a$) and Crossover Region ($T = T_a$) of the Polymer System[a]

b	$\gamma_1(T > T_a)$	$\gamma_{11}(T > T_a)$	$\gamma_1(T = T_a)$	$\gamma_{11}(T = T_a)$	$\gamma_s(T \geqslant T_a)$
2	0.7661	−0.3102	1.3637	0.8850	1.8424
	0.7659 ± 0.0022	-0.3102 ± 0.0016	1.3632 ± 0.0147	0.8844 ± 0.0266	1.8420 ± 0.0029
3	0.8903	−0.1608	1.4211	0.9008	1.9414
	0.8908 ± 0.0022	-0.1602 ± 0.0018	1.2413 ± 0.0066	0.9008 ± 0.0107	1.9417 ± 0.0026
4	0.9815	−0.0408	1.4578	0.9118	2.0038
	0.9829 ± 0.0023	-0.0405 ± 0.0020	1.4594 ± 0.0052	0.9126 ± 0.0078	2.0062 ± 0.0026
5	1.0544	0.0566	1.4877	0.9232	2.0522
	1.0549 ± 0.0024	0.0562 ± 0.0022	1.4903 ± 0.0047	0.9271 ± 0.0068	2.0535 ± 0.0027
6	1.1142	0.1361	1.5139	0.9355	2.0923
	1.1161 ± 0.0026	0.1368 ± 0.0023	1.5171 ± 0.0046	0.9387 ± 0.0064	2.0954 ± 0.0028
7	1.1662	0.2048	1.5380	0.9486	2.1275
	1.1687 ± 0.0027	0.2078 ± 0.0025	1.5398 ± 0.0046	0.9500 ± 0.0063	2.1296 ± 0.0029
8	1.2106	0.2619	1.5605	0.9617	2.1592
	1.2135 ± 0.0024	0.2655 ± 0.0019	1.5633 ± 0.0046	0.9650 ± 0.0062	2.1615 ± 0.0030
9	1.2497 ± 0.0029	0.3135 ± 0.0028	1.5793 ± 0.0047	0.9726 ± 0.0063	2.1859 ± 0.0031

10	1.2887 ± 0.0029	0.3620 ± 0.0028	1.6013 ± 0.0046	0.9872 ± 0.0063	2.2153 ± 0.0029
12	1.3459 ± 0.0033	0.4320 ± 0.0032	1.6361 ± 0.0051	1.0125 ± 0.0067	2.2597 ± 0.0035
15	1.4175 ± 0.0037	0.5152 ± 0.0035	1.6824 ± 0.0056	1.0450 ± 0.0073	2.3198 ± 0.0039
17	1.4582 ± 0.0038	0.5637 ± 0.0037	1.7076 ± 0.0057	1.0624 ± 0.0075	2.3528 ± 0.0040
20	1.5133 ± 0.0042	0.6235 ± 0.0041	1.7494 ± 0.0062	1.0958 ± 0.0081	2.4030 ± 0.0042
22	1.5520 ± 0.0045	0.6608 ± 0.0044	1.7818 ± 0.0066	1.1205 ± 0.0086	2.4431 ± 0.0047
26	1.6056 ± 0.0049	0.7205 ± 0.0048	1.8165 ± 0.0072	1.1424 ± 0.0094	2.4907 ± 0.0051
30	1.6489 ± 0.0055	0.7744 ± 0.0053	1.8442 ± 0.0080	0.1649 ± 0.0104	2.5235 ± 0.0056
35	1.7101 ± 0.0061	0.8324 ± 0.0059	1.8949 ± 0.0089	1.2029 ± 0.0115	2.5878 ± 0.0063
40	1.7451 ± 0.0065	0.8678 ± 0.0064	1.9199 ± 0.0095	1.2173 ± 0.0124	2.6224 ± 0.0066
50	1.8187 ± 0.0078	0.9412 ± 0.0078	1.9790 ± 0.0076	1.2617 ± 0.0149	2.6963 ± 0.0078
60	1.8753 ± 0.0087	0.9966 ± 0.0088	2.0199 ± 0.0127	1.2857 ± 0.0167	2.7540 ± 0.0087
70	1.9116 ± 0.0098	1.0392 ± 0.0095	2.0505 ± 0.0142	1.3170 ± 0.0182	2.7839 ± 0.0102
80	1.9755 ± 0.0113	1.1134 ± 0.0105	2.0945 ± 0.0172	1.3513 ± 0.0224	2.8376 ± 0.0120

[a] Number of MC simulations for each entry was 500,000. The agreement is seen to be exceptionally good for exact RG and MCRG findings for $2 \leqslant b \leqslant 8$.

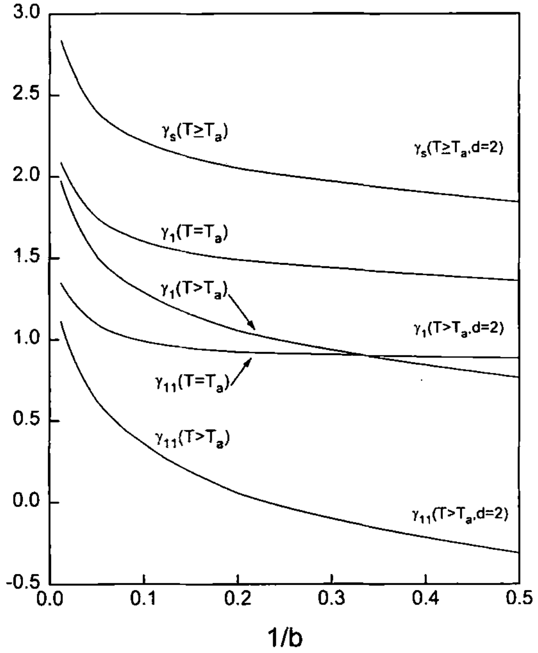

Fig. 2. The full lines represents the dependence of the critical exponents γ_1, γ_{11}, and γ_s in the bulk phase ($T > T_a$) and the crossover region ($T = T_a$) as a function of $1/b$ ($2 \leqslant b \leqslant 80$). The horizontal dotted lines correspond to the known two-dimensional Euclidean values $\gamma_1 = 61/64$ and $\gamma_{11} = -3/16$ for the bulk region ($T > T_a$) and $\gamma_s = 67/32$ for both the bulk phase and the crossover region ($T \geqslant T_a$).

At the end, one may pose the question about the possible asymptotic behavior of the critical exponents γ_1, γ_{11}, and γ_s in the limit $b \to \infty$ (when the SG fractal dimension approaches the Euclidean value 2). This question has been answered[9] for the critical exponents γ and ν (within the finite-size scaling approach), with the predictions $\gamma \to 133/32$ and $\nu \to 3/4$ when $b \to \infty$. Combining these predictions with the scaling relation (22), one finds[2] that γ_s tends to 157/32 from below. As regards the other two critical exponents, that is, γ_1 ad γ_{11}, their asymptotic behavior cannot be obtained in a similar indirect way, but it should be reached by making an independent finite-size scaling approach (in both the bulk phase and crossover region), which will be a matter for future study.

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